Discrete Mathematics
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Lecture -16
Relations

(Refer Slide Time: 00:24)

Lecture Overview

❑ Relations

❖ Various definitions and properties

❖ Representation of relations

❖ Special types of relations

Hello everyone, welcome to this lecture on relations. Just to recap, in the last lecture we introduced the definition of sets and various set theoretic operations, we also saw various set theoretic identities. The plan for this lecture is as follows. In this lecture we will introduce what we call as a relations, we will see their various properties, we will discuss how to represent relations and we will see some special types of relations.

(Refer Slide Time: 00:48)

Relations

❑ Consider the following table (database)

c_1	c_2
Afghanistan	Kabul
India	New Delhi
Egypt	Cairo
England	London

Table T

❑ The elements of the first and second column are related"

❖ Countries and their capitals

❑ Mathematical interpretation of T ?

❖ A: set of countries in the world

❖ B: set of cities in the world

❖ T: a subset of $A \times B$

❑ The above concept can be extended to more than one set

So, what is a relation? So, let me begin with this example, you consider this table and people who are familiar with databases, they very well know what is a table. Table basically consist of several columns and each column have some entries. So, I call this table T which has two columns, column number 1 and column number 2. In column number 1, you have some countries and in column number 2, you have some cities listed.

Now if your general knowledge is good then it turns out that the elements you can spot here that elements of the first column and the second column they are related by some relationship. And the relationship here is that, in the first column I have listed some of the countries and in the second column I have listed the capital of the corresponding countries. Of course, I can add multiple entries in this table and I am not doing that.

Now how do we mathematically interpret this table? Is there any mathematical interpretation or mathematical abstraction by which you can define this table? Well the way I can mathematically interpret this table T is as follows. I can imagine that I have a set A which is defined to be the set of all countries in the world. Well I do not know right now how many exact numbers of countries in the world definitely it is more than 200.

So, A has more than 200 elements. Whereas B is another set, which is defined to be the set of all cities in the world. Again this is a well defined set because we know the list of all cities in this

world, so both these sets are well defined. Now if I take the Cartesian product of A and B, what will I obtain? The Cartesian product of A and B will be a set of the form (a, b), where a will be some country.

Namely, it will be belonging to A and b will be some city. As of now when I take A x B, there is no relationship between the elements a,b, I am just picking some country and some city, country, city. I have listed down all possible pairs of the form country, city and this will be an enormously large set. Now if I take a subset of that A x B, a special subset of that A x B and call it T and what is that special subset?

Namely I take a subset where the 'a' component is a country and 'b' component will be the capital. And if I take only components or pairs of the form (a,b) of this form, where a is the country and b is the capital and list down all such (a, b) pairs I obtain this table T. So, you can imagine that this table T here is nothing but a special subset of A x B. So, to demonstrate, my A x B could also consist of elements of the form Afghanistan, New Delhi, it will also have elements of the form India, Cairo and so on. I am not taking those elements, I am taking only those elements (Afghanistan, Kabul) from A x B, I am taking only the element (India, New Delhi) and so on. So, I am taking a subset of A x B and only those subsets which have a special relationship among the, a component and the b component.

So, that is a loose definition of a relation. A relation here is basically a subset of A x B, if I am considering two sets A and B and of course whatever I have discussed here can be extended where I have multiple sets. What do I mean by that? In this example I had only two columns, C_1 C_2. And C_1 was having entries from set A and C_2 was having some entries from B. What if I have a database consisting of 3 columns?

Say there is a third column as well, where the third column denotes population. So, those entries will be coming from a set C, what if I have a fourth column which denotes another feature of the table, say the climate or the temperature of the respective countries. So, those entries will be coming from another set D and so on. And the table with some specific entries will be considered as a subset of the Cartesian product of all the big sets A, B, C, D from which the elements in

your column C_1, C_2, C_3, C_4 are occurring here.

(Refer Slide Time: 06:17)

Relations

❑ Binary relations

   ❖ Let A and B be two sets (need not be different)

   ❖ A binary relation R from A to B is a subset of $A \times B$ --- order matters

❑ Notations:

   ❖ aRb if $(a,b) \in R$

   ❖ $a \not R b$: if $(a,b) \notin R$

❑ A relation can be defined from a set to itself

   ❖ Let $A = \{1,2,3,4\}$ and $R = \{(a,b): a \text{ divides } b\}$

   ❖ $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$

$R \subseteq A \times A$

$\{(1,1),(1,2),(1,3),(1,4),$
$(2,1),(2,2),(2,3),(2,4)$
$(3,1),(3,2),(3,3),(3,4)$
$(4,1),(4,2),(4,3),(4,4)\}$

$R': \{(a,b): a<b\}$

So, that is how we are going to define a relation, so we will focus in mostly on binary relations and by binary relations I mean, we will be working with two sets A and B, but whatever we are discussing here can be generalized for extended for any number, it can be generalized for n-ary relations which are defined over n sets. But for this course and for most important cases, we will be focusing on binary relations.

So, how do we define a binary relation? So, we are given two sets here, call them A and B and they need not be different, I stress here, they can be the same, definition does not say that they have to be different sets, because we are defining a relation in an abstract fashion. Then a binary relation from A to B is a subset of A x B. So, I have highlighted the term from A to B by a different color because the order of the relation matters.

So, if you are defining a relation from A to B then the relation should be a Cartesian product of A x B. Whereas if your relation is from B to A, then that relation should be a Cartesian product of B x A or it should be Cartesian product of B and A, it should be a subset of B x A. So, the order matters here a lot. Now when I say a subset my relation R could be empty as well, that means my table could have zero entries, that is also possible, that is also a valid table.

So, it is not necessary that a relation always should have at least one element of the form (a,b) it could be empty as well. So, we use some notations here when dealing with relations. So if the element (a, b) belongs to the relation R then I will be often writing this expression aRb, I will be saying a followed by capital R followed by b, to denote that a is related to b whereas if the element (a,b) does not belong to the relation R, then I will strike off R in this expression. So, as I said here when I am defining the binary relation my sets A and B could be the same, they could be different. So, I can define a relation from the set to itself so let me demonstrate this. So, imagine I have a set A consisting of the elements {1, 2, 3, 4} and I define a relation R consisting of all elements of the form (a, b) we are both a and b are from the set A such that a divides b.
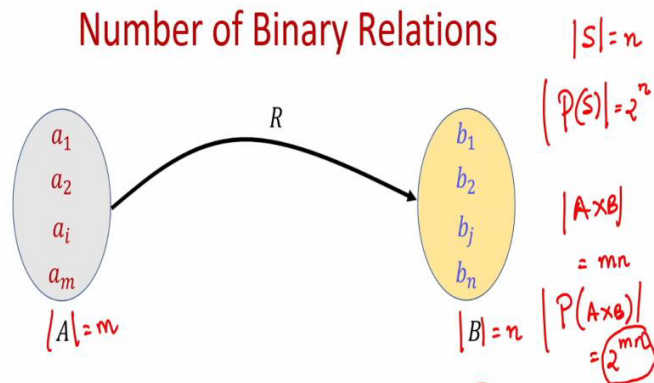
So, basically I am defining a relation over A x A. So A x A here will have many elements here, it will have (1, 1) it will have (1, 2), (1, 3), (1, 4). It will have the elements (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4) and (4, 1), (4, 2), (4, 3), (4, 4). So, I will be taking only those components, (a, b) from this collection A x A where the first component a divides the second component b.

So, it turns out that I will be taking only (1, 1) because one divides one. I will be taking (1, 2), I will be taking (1, 3), I will be taking (1, 4). But I would not be taking (2, 1) in the relation R, because for (2, 1), a is 2, b is 1 and 2 does not divide 1. I will be taking (2, 2) but I would not be taking (2, 3) because 2 does not divide 3, I will be taking (2, 4) and so on. So, the elements of the relation are listed down here.

I could define another relation, I could define a relation R' collect consisting of all (a, b) such that a < b. That is another relation and that will be consisting of other pairs, it might be consisting of other pairs different from the pairs which you have listed down in R. So, it depends upon the property which you want to be satisfied by the elements of the relation that defines that tuples or the pairs which will be present in that relation.

(Refer Slide Time: 11:02)

Number of Binary Relations

$|S| = n$

$\left(|P(S)| = 2^n\right)$

$|A \times B|$

$= mn$

$\left|P(A \times B)\right|$

$= 2^{mn}$

$A \times B = \{(a, b):$

$m \ (a) \in A$

$n \ (b) \in B\}$

$R \subseteq A \times B$

$|A| = m$

$|B| = n$

❑ How many relations possible from A to B ?

❖ # of relations from A to $B = |P(A \times B)| = 2^{mn}$

❖ Every relation from $\underline{A}$ to B is a subset of $(A \times B)$

So, now an interesting question is that if you have a binary relation defined from the set A to B how many such binary relations can you define? Can I define any number of binary relations or is there an upper bound on the maximum number of binary relations that I can define? Namely how many tables I can form with two columns, where the first column can take entries from the set A and a second column can take entries from the set B.

Well it turns out that I can form 2^{mn} number of binary relations provided A has m number of elements and B has n number of elements. And this simply comes from the observation that you take any relation R, it is nothing but subset of A x B. So, you have this bigger set A x B consisting of all elements of the form (a, b) where a is from A and b is from B, you pick any subset of this A x B that gives you a relation.
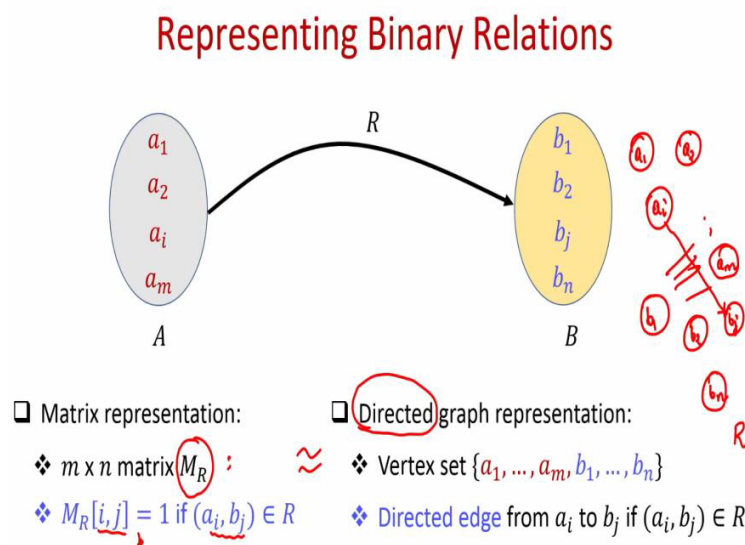
So, the number of binary relations is nothing but how many different subsets of A x B you can form. Namely the cardinality of the number of relations will be the same as the cardinality of the power set of A x B and what will be the power set of A x B, so you recall the theorem that we proved in the last lecture. The last lecture we proved that if the cardinality of a set S is n, then the cardinality of the power set is 2^n.

There will be 2^n subsets of a set consisting of n elements. Now what will be the cardinality of A x B? Well the cardinality of A x B will be m times n. Why? Because it is the collection of all (a,

b) pairs, where a is from A. So, how many different values of a you can have m and how many different values of b you can have, n. So, you have m times n number of (a,b) pairs, which will be present in A x B.

So, your cardinality of A x B is mn and that means the number of subsets of A x B that you can form is 2^{mn}. That means these 2^{mn} is the maximum number of tables that you can form with 2 columns, where the first column takes entries from the set A and the second column takes the entries from the set B.

(Refer Slide Time: 14:06)

Representing Binary Relations

- Matrix representation:
  - $m \times n$ matrix M_R : $\approx$
  - $M_R[i,j] = 1$ if $(a_i, b_j) \in R$

- Directed graph representation:
  - Vertex set $\{a_1, ..., a_m, b_1, ..., b_n\}$
  - Directed edge from a_i to b_j if $(a_i, b_j) \in R$

Now the next question is how do we represent binary relations? So, there are some well known methods for representing binary relations, the first method is the matrix representation. So, since we are dealing with binary relations the matrix representation here will be an m x n matrix, m because there are m possible elements from the set A and n columns because I have n possible elements from the set B.

And I will be denoting this matrix by M_R, where R is the relation. And what will be the entries of this matrix. So, it will be a Boolean matrix where the entry number ith row and jth column will be 1 provided the element (a_i, b_j) is present in the relation R. So, what I am saying here is that imagine you have constructed a table or a relation R, so it will be either consisting of 0 number of rows or some number of rows and each row will have 2 columns.

So, if you have a row where an element a_i is present in the first column and b_j is present in the second column, that means you have (a_i, b_j) present in the relation R, then in the matrix representation what you will do is you will go to the ith row, j^{th} column and you will put an entry 1. Whereas if the entry number (a_i, b_j) is not there in your database or in the relation R if it is not present, then the ith row and j^{th} column entry will become 0.

That means you want to denote there that element a_i is not related to the element b_j as per your relation R. So, that is the matrix representation, so if I want to now look back into the question that how many binary relations I can form, well it turns out to the answer for how many different or distinct Boolean matrices of dimension m x n you can construct because each Boolean matrix will correspond to one binary relation.

You cannot have two different relations or two different tables represented by the same Boolean matrix that is not going to happen. So, this is the matrix representation for representing a relation, we have another representation which we call as the directed graph representation. So, what do we do in this representation, we draw a graph and by graph I mean a collection of vertices and edges.

The vertices will be the nodes a_1, a_2, a_m, and b_1, b_2, b_n and it will be a directed graph that means the edges here will have a direction associated, how the edges are added in this graph? So, if this graph represents the relation R, and if in the relation R the element a_i is related to the element b_j, that means in the database of R, you have a row with first column being a_i and the second column being b_j, that means a_i is related to b_j.

Then what you will do is you will take the node a_i here and b_j here and you will add a directed edge from a_i to b_j, so the edge here denotes that a_i is related to b_j. The direction here matters, if the edge from a_i to b_j is present that does not mean that the edge from b_j to a_i is also present. That depends whether b_j is related to a_i or not, you might have a relation where only a_i is related to b_j.
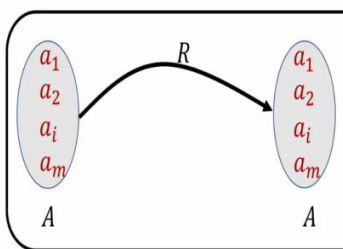
But b_j is not related to a_i in which case the reverse direction edge may not be present. And now

you can see here that the matrix representation and the directed graph representation they are equivalent to each other. If you have the entry a_i, b_j in the matrix representation 1, then you will have the edge from a_i to b_j, whereas if in the ith row and j^{th} column of the matrix, you have the entry 0, then this edge from a_i to b_j will not be present.

You might be wondering why we have two different representations. We will be seeing that depending upon how we want to prove or whether what kind of properties we want to prove regarding the relations, the representation matters a lot. There might be cases where if we use the matrix representation then the arguments become very simple, whereas there might be cases where we want to deal with the directed graph representation.

(Refer Slide Time: 19:19)

Now, we would define some special types of relations. So, the first special type of relation is a reflexive relation, it is a relation defined from the set to itself. And when do we say that the relation is reflexive, as the term reflects here signifies, the relation will be called a reflexive if every element from the set A is related to itself as per the relation. That means you take any element from the set A, the element A should be related to itself as per the relation.

If this is true for every element a, from the set A then I say that my relation is reflexive. Even if there is one a, one element a, which is not related to itself in the relation, then it will not be considered as a reflexive relation. So, now if I want to interpret the matrix representation of a

reflexive relation, it is easy to see that its relation R is reflexive. Then the matrix M_R will be n x n matrix.

Because the relation is defined from the set to itself and if it is reflexive then all the diagonal entries will be 1. Because I will be having (a_1, a_1) present in the relation, I will be having (a_2, a_2) present in the relation and in the same way I will be having (a_n, a_n) also present in the relation. So, since (a_1, a_1) is present that means the entry (1, 1) will be 1, (a_2, a_2) is present that means the entry (2, 2) in the matrix will be 1.

Since the a_n is related to a_n that means the entry number (n,n) in the matrix will be 1 which is equivalent to saying that all the diagonal entries will be 1. There might be additional entries in the relation apart from these reflexive entries that is also fine. The definition of the reflexive relation says that you only want the guarantee that every element from the set A should be related to itself as per the relation.

There might be other elements which are related to the elements of A as per the relation R, I do not care about those elements. If I focus on the graph representation of reflexive relation, then it will be a special type of graph where I will be having a self-loop at each node of the graph. Because a_1 is related to itself, that means I will be having a loop or a directed edge from a_1 to a_1. Since a_2 is related to itself, I will be having an edge from a_2 to itself and so on.

So, now let us see some examples of reflexive relations. So, I have a set here A consisting of 2 elements {1, 2} and I have given you many relations over this set. We have to find out which of these relations are reflexive relations. So, let us start with the relation R_1. It is reflexive because indeed the element (1, 1) is present in the relation and the element (2, 2) is present in the relation. So, this satisfies the definition of reflexive relation.

My relation R_2, it also satisfies the definition of reflexive relation because (1, 1) is present (2, 2) is present and I have (1, 2) is also present here, but that is fine because even the element (1, 2) does not violate the truth of this universal quantification for the relation R_2. What is this universal quantification? It says that if 1 is present in the set A then (1, 1) should be in your

relation.

If 2 is present in your set A then (2, 2) should be present in your relation. That is all, it does not say whether anything additional is present or not. Now I come to the relation R_3, my relation R_3 is not a reflexive relation, because (2, 2) is not present here. That means if I take this definition of reflexive relation here, this universal quantification is not true, because indeed I have an element 2 in my set A, and 2 is present in A, but (2, 2) is not present in R_3, this is not happening. That means this universal quantification is false and that is why R_3 is not an example of reflexive relation and what about R_4? Again, R_4 is not a reflexive relation over A. Because I have 1 present in A, but (1, 1) is not present in R_4. If I consider this implication, this implication is false because 1 is present in the set A, but (1, 1) is not present in R_4.

Because R_4 is empty in the same way 2 is present in A, but (2, 2) is not present in R_4, that means this implication is also false. And since both the implications are false here this universal quantification for this relation R_4 is not true. That is why relation R_4 is not an example of reflexive relation. Now here is an interesting question for you: can it happen that the set ϕ or the relation ϕ is a reflexive relation over a set A.

So, remember ϕ is also a relation because ϕ is a subset of Cartesian product of any A x A. So, the question here is: is it possible that ϕ is a valid reflexive relation over some set A? Might look no, but the answer is yes. If A is empty by itself then ϕ is a valid reflexive relation over A. This is because if A is ϕ, then A x A will also be ϕ. And the universal quantification which is there in the definition of reflexive relation will be true for the relation R equal to ϕ.

So, you take R equal to ϕ, so remember ϕ is always a valid subset of A x A and for the relation R equal to ϕ this universal quantification is true for a being ϕ, because the definition says for every a belonging to ϕ. But no element belongs to ϕ, so vacuously this implication will be true for the relation R equal to ϕ. So, we can have ϕ as a valid reflexive relation provided the set over which the relation is defined is an empty set. But if A over which the relation is defined is non empty, then ϕ can never be a valid reflexive relation.

(Refer Slide Time: 27:31)